A FINITENESS THEOREM IN THE GALOIS COHOMOLOGY
OF ALGEBRAIC NUMBER FIELDS

WAYNE RASKIND

Dedicated to the memory of Professor D. S. Rim

ABSTRACT. In this note we show that if k is an algebraic number field with algebraic closure $\bar{k}$ and M is a finitely generated, free $\mathbf{Z}_l$-module with continuous $\mathrm{Gal}(\bar{k}/k)$-action, then the continuous Galois cohomology group $H^1(k, M)$ is a finitely generated $\mathbf{Z}_l$-module under certain conditions on M (see Theorem 1 below). Also, we present a simpler construction of a mapping due to S. Bloch which relates torsion algebraic cycles and étale cohomology.

The Galois cohomology of an algebraic number field is well-known to be important in the study of the arithmetic of the field and of algebraic varieties defined over the field. Unfortunately, this cohomology is not always well-behaved; for example, the Galois cohomology group

$$H^1\big(k, \mathbf{Z}_l(1)\big) =: \varprojlim_n H^1\big(\mathrm{Gal}(\bar{k}/k), \mu_{l^n}\big)$$

($\bar{k}$ an algebraic closure of k) is not a finitely generated $\mathbf{Z}_l$-module. However, there are many important cases in which one knows that the group $H^1(k, M)$ is a finitely generated $\mathbf{Z}_l$-module when M is a finitely generated $\mathbf{Z}_l$-module. In this note we prove a theorem to this effect when M is the l-adic étale cohomology group $H^i(\bar{X}, \mathbf{Z}_l(j))$ modulo torsion of the extension to $\bar{k}$ of a smooth, proper algebraic variety defined over k and $i \neq 2j, 2(j-1)$. One may view this theorem as a generalization of the weak Mordell-Weil Theorem for the k-points of an abelian variety; indeed, when M is the Tate module of such a variety our theorem basically comes down to the weak Mordell-Weil Theorem. Also, this theorem gives a much simpler proof of the weak Mordell-Weil Theorem for $H^0(X, K_2)/K_2 k$ as proved in [7].

We note that a stronger version of our main theorem was proven in the thesis of P. Schneider (see Remark 2 following Theorem 1 for more details). Our excuses for publishing this note are first, that Schneider's result has not been published and second, that our proof is quite short and self-contained. We hope that the applications of the theorem will justify this redundancy.

In §2 we present a new construction of a mapping due to Bloch which relates torsion algebraic cycles and étale cohomology. This simplifies the original construction considerably and explains the relation between this mapping and another mapping relating algebraic cycles and étale cohomology which was also defined by Bloch [3, Corollary 1.2].

I should like to thank S. Bloch and J.-L. Colliot-Thélène for listening to the proof of Theorem 1, and M. Levine for convincing me to write up §2.

0. Notation. X will denote an algebraic variety over a field k, $\bar{k}$ will denote a separable closure of k and $G = \mathrm{Gal}(\bar{k}/k)$ with the Krull topology. l is a rational prime different from the characteristic of k and μ_{l^n} will denote the étale sheaf (Galois module) represented by the étale k-group scheme μ_{l^n}. This defines an étale sheaf on the k-variety X. For $j \in \mathbf{Z}$, $\mu_{l^n}^{\otimes j}$ denotes the étale sheaf obtained by twisting μ_{l^n} j-times via the cyclotomic character.

When k is an algebraic number field, v will denote a place of k and k_v the completion of k for the v-adic valuation. k_v^{nr} will denote the maximal unramified extension of k_v in a fixed $\bar{k}_v$. The inertia group $\mathrm{Gal}(\bar{k}_v/k_v^{nr})$ will be denoted by I_v and Γ_v will denote $\mathrm{Gal}(k_v^{nr}/k_v)$.

By $C^n(X)$ we shall mean the group of codimension n algebraic cycles on X. We have the following subgroups of $C^n(X)$, $C_{\mathrm{hom}}^n, C_{\mathrm{alg}}^n, C_{\mathrm{rat}}^n$, which denote respectively the subgroup of cycles homologically equivalent to zero over $\bar{k}$, algebraically equivalent to zero and rationally equivalent to zero. $CH^n(X) =: C^n(X)/C_{\mathrm{rat}}^n(X)$ is the nth Chow group of X. Finally, the notation H_{ct}^i denotes continuous Galois cohomology in the sense of [13].

1. The main theorem.

THEOREM 1. *Let k be an algebraic number field and M a finitely generated free $\mathbf{Z}_l$-module upon which $\mathrm{Gal}(\bar{k}/k)$ acts continuously. Assume that both M and $M(-1) =: M \otimes_{\mathbf{Z}_l} \mathbf{Z}_l(-1)$ (with diagonal G-action) satisfy the following conditions:*

(1) For almost all finite places v of k, the inertia group I_v acts trivially on M.

(2) For almost all finite places satisfying (1), no eigenvalue of the geometric Frobenius F_v acting on $M \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$ is a root of unity.

Then $H_{ct}^1(k, M)$ is a finitely generated $\mathbf{Z}_l$-module.

PROOF. By [13, Corollary 2.1], it will be enough to show that $H_{ct}^1(k, M)/l$ is finite. Since M is torsion free there is an exact sequence of G-modules ($G = \mathrm{Gal}(\bar{k}/k)$):

$$0 \to M \to M \to M/l \to 0.$$

Taking continuous cohomology then gives an injection

$$H^1(k, M)/l \hookrightarrow H^1(k, M/l).$$

To ease notation we shall drop the subscript ct.

We will show that there is a finite set S of places of k such that

$$H^1(k, M)/l \hookrightarrow \mathrm{Ker}\left[H^1(k, M/l) \to \prod_{v \notin S} H^1(k_v^{nr}, M/l) \right].$$

Since this last group is finite [10, Chapter 11, §6, Théorème 7], this will prove the theorem.

LEMMA 1. *Let S be a finite set of places of k containing the archimedean places, the places dividing l and those places for which M or $M(-1)$ does not satisfy condition* (1) *or* (2). *Then for $v \notin S$ we have*

$$\varinjlim_{k_v \subset L \subset k_v^{nr}} \left[H^1(L, M)/l \right] = 0.$$

Here the transition maps are given by restriction.

REMARK. The proof of the lemma is the heart of the proof of the theorem.

PROOF OF LEMMA. Let v be a place not in S and let L be an unramified extension of k_v. Denoting $\Gamma_L = \mathrm{Gal}(k^{nr}/L)$ and $I = \mathrm{Gal}(\overline{k}_v/k_v^{nr})$ and recalling that $\Gamma_L \cong \hat{\mathbf{Z}}$ is of cohomological dimension one for torsion modules, we have an exact sequence

$$0 \to H^1(\Gamma_L, M/l^n) \to H^1(L, M/l^n) \to H^1(I, M/l^n)^{\Gamma_L} \to 0.$$

Since all of the groups in this exact sequence are finite we may safely pass to the inverse limit over n to obtain the exact sequence

$$(*) \qquad 0 \to H^1(\Gamma_L, M) \to H^1(L, M) \to H^1(I, M)^{\Gamma_L} \to 0.$$

Now, I, considered as a Γ_L-module, is an extension of $\prod_{p \neq p_v} \mathbf{Z}_p(1)$ by a pro p_v-group R, where p_v denotes the residue characteristic of k_v. We have chosen v such that the action of I on M is trivial and $l \neq p_v$. Hence $H^1(I, M) \cong \mathrm{Hom}(\mathbf{Z}_l(1), M) \cong M(-1)$. But condition (2) and the fact that M is torsion free ensure that $M(-1)^{\Gamma_L} = 0$ and hence the right-hand side of $(*)$ is zero.

For the left-hand side we have, since $(M \otimes_{\mathbf{Z}_l} \mathbf{Q}_l)^{\Gamma_L} = 0$,

$$H^1(\Gamma_L, M) \cong \left(M \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l \right)^{\Gamma_L}.$$

Passing to the direct limit over all unramified L/k_v we get

$$\varinjlim_{k_v \subset L \subset k_v^{nr}} \left(M \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l \right)^{\Gamma_L} = \left(M \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l \right)^I = M \otimes \mathbf{Q}_l/\mathbf{Z}_l.$$

Hence

$$\varinjlim_{k_v \subset L \subset k_v^{nr}} \left[H^1(L, M)/l \right] = \left[\varinjlim_{k_v \subset L \subset k_v^{nr}} H^1(L, M) \right] \Big/ l = (M \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l)/l = 0.$$

This completes the proof of the lemma.

REMARK. What seems to make the proof of the lemma work is that continuous cohomology with l-adic coefficients does not commute with direct limits!

COMPLETION OF THE PROOF OF THEOREM 1. We have a commutative diagram:

$$
\begin{array}{ccc}
H^1(k, M)/l & \hookrightarrow & H^1(k, M/l) \\
\downarrow & & \downarrow \\
\prod_{v \notin S} \varinjlim_{k_v \subset L \subset k_v^{nr}} H^1(L, M)/l & \to & \prod_{v \notin S} \varinjlim_{k_v \subset L \subset k_v^{nr}} H^1(L, M/l)
\end{array}
$$

By Lemma 1, the bottom left group vanishes and since cohomology with *finite* coefficients commutes with direct limits, we have

$$\varinjlim_{k_r \subset L \subset k_v^{nr}} H^1(L, M/l) = H^1(k_v^{nr}, M/l).$$

Hence,

$$H^1(k, M)/l \hookrightarrow \ker\left[H^1(k, M/l) \to \prod_{v \notin S} H^1(k_v^{nr}, M/l) \right]$$

as claimed. This completes the proof of Theorem 1.

EXAMPLES. (1) Let $M = \mathbf{Z}_l(j)$ for $j \neq 0$ or 1. Then M satisfies the hypotheses of the theorem and so $H^1(k, \mathbf{Z}_l(j))$ is a finitely generated $\mathbf{Z}_l$-module. This has been proven by P. Schneider [8, Lemma 5, (iii)] and Soulé shows [11, Statement b), p. 376] that for $j > 1$,

$$\mathrm{rk}_{\mathbf{Z}_l} H^1(k, \mathbf{Z}_l(j)) = \begin{cases} r_2 & \text{if } j \text{ is even,} \\ r_1 + r_2 & \text{if } j \text{ is odd.} \end{cases}$$

In particular this rank is independent of l.

Note that $H^1(k, \mathbf{Z}_l)$ is a finitely generated $\mathbf{Z}_l$-module (as can be deduced from class field theory) but our theorem does not apply to this case. That $\mathrm{rk}_{\mathbf{Z}_l} H^1(k, \mathbf{Z}_l)$ should be equal to $1 + r_2$ is one form of the Leopoldt conjecture.

(2) (It is this example which motivated the theorem.)

Let X be a smooth, proper, geometrically connected variety over a number field k and let

$$\overline{X} = X \times_k \overline{k}.$$

Then for $i \neq 2j$, $2(j-1)$, the $\mathbf{Z}_l$-module $H^i(\overline{X}, \mathbf{Z}_l(j))/$ torsion satisfies the hypotheses of the theorem. This follows from the theory of étale cohomology and the Weil conjectures as proved by Grothendieck and Deligne (cf. [4, p. 781] and [5, Theorem 1]). However, if M is the Tate module of an abelian variety over k then M satisfies conditions (1) and (2) of the theorem without appeal to the Riemann Hypothesis for étale cohomology (I thank J. Tate for pointing this out to me). In this case the theorem is none but the weak Mordell-Weil Theorem [6].

The condition $i \neq 2j$ is unnecessary as will be seen in Remark (2) below.

REMARKS. (1) Theorem 1 now gives us a simple proof of the "weak Mordell-Weil Theorem" for $H^0(X, K_2)/K_2 k$ as proved in [7, Théorème 3]. Indeed, the exact sequence of Suslin [12, Corollary 4.4])

$$0 \to H^0(X, K_2)/l^n \to H^2(X, \mathbf{Z}/l^n(2)) \to {}_{l^n}H^1(X, K_2) \to 0$$

and the main theorem of [13] and a Mittag-Leffler argument give us an injection

$$\varprojlim_n \left[H^0(X, K_2)/K_2 k \right]/l^n \hookrightarrow H^2(X, \mathbf{Z}_l(2))/H^2(k, \mathbf{Z}_l(2)),$$

and it follows easily from the Hochschild-Serre spectral sequence, another Mittag-Leffler argument and Theorem 1 that the group on the right is a finitely generated $\mathbf{Z}_l$-module. Hence $[H^0(X, K_2)/K_2 k]/l$ is finite and it follows easily that for any positive integer m, $[H^0(X, K_2)/K_2 k]/m$ is finite.

(2) In [**9**, §4, Satz 5 and Satz 6] Schneider proves the following more precise and general version of Theorem 1: Let M be an almost everywhere unramified finitely generated free $\mathbf{Z}_l$-module with continuous $\mathrm{Gal}(\bar{k}/k)$ action. Set $N = M \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l$ and $\tilde{M} = \mathrm{Hom}(N(-1), \mathbf{Q}_l/\mathbf{Z}_l)$. Then the maximal divisible subgroup of $H^1(k, N)$ is of finite cotype iff $H^0(k_v, \tilde{M}) = 0$ for almost all finite places v of k. If this is the case and $d = \mathrm{corank}$ of $H^1(k, N)$, then

$$d \geqslant \mathrm{rank}_{\mathbf{Z}_l} H^0(k, M) + \sum_{v \text{ finite}} \mathrm{rank}_{\mathbf{Z}_l} H^0(k_v, \tilde{M}).$$

Thus Schneider's result applies to the case $i = 2j$ in Example 2 above.

2. On Bloch's map λ_l^n. In [**2**, §2], Bloch constructs a mapping relating the l-primary torsion of $CH^n(X)$ with an l-primary étale cohomology group $H^{2n-1}(X, \mathbf{Q}_l/\mathbf{Z}_l(n))$ for X smooth and proper over a separably closed field k of characteristic different from l. The construction of λ_l^n uses very deep results such as the Weil conjectures and the theorems of Bloch and Ogus [**1**]. In this section we construct a mapping relating torsion in Chow groups to étale cohomology for any smooth variety over any field k. The construction does not use the Weil conjectures or the results of Bloch and Ogus. However, if X is proper over k, the Weil conjectures may be used to show that our mapping goes between essentially the same two groups as Bloch's mapping. We also point out that the method used to construct our map is also essentially due to Bloch [**3**].

Let X be a smooth algebraic variety over a finitely generated field k, $l \neq \mathrm{char}\, k$ a prime number. Let $\bar{k}$ be a separable closure of k and $\bar{X} = X \times_k \bar{k}$. The cycle map [**14**, [cycle], 2.2.10, p. 143] and the Hochschild-Serre spectral sequence give a commutative diagram (see §0 for notation):

$$
\begin{array}{ccccccccc}
0 & \to & C_{\mathrm{hom}}^n/C_{\mathrm{rat}}^n(X) & \to & CH^n(X) & \to & C^n/C_{\mathrm{hom}}^n(X) & \to & 0 \\
& & \downarrow f_m & & \downarrow & & \downarrow & & \\
0 & \to & E_1^{2n}(X) & \to & H^{2n}(X, \mathbf{Z}/l^m(i)) & \to & H^{2n}(\bar{X}, \mathbf{Z}/l^m(i))^G & &
\end{array}
$$

which induces the left vertical arrow. From the spectral sequence we get a map

$$E_1^{2n}(X) \xrightarrow{g_m} H^1(k, H^{2n-1}(\bar{X}, \mathbf{Z}/l^m(n))).$$

We define our map

$$\rho_l^n: C_{\mathrm{hom}}^n/C_{\mathrm{rat}}^n(X) \to H^1(k, H^{2n-1}(\bar{X}, \mathbf{Z}_l(n)))$$

to be the inverse limit over m of the composites $g_m f_m$. To examine ρ_l^n on torsion assume for the moment that there is no torsion in $H^{2n-1}(\bar{X}, \mathbf{Z}_l(n))$ and $H^{2n}(\bar{X}, \mathbf{Z}_l(n))$. Then using the fact that $H^{2n-1}(\bar{X}, \mathbf{Z}/l^m(n))$ is finite we may construct an exact sequence of topological G-modules ($G = \mathrm{Gal}(\bar{k}/k)$):

$$0 \to H^{2n-1}(\bar{X}, \mathbf{Z}_l(n)) \to H^{2n-1}(\bar{X}, \mathbf{Q}_l(n)) \to H^{2n-1}(\bar{X}, \mathbf{Q}_l/\mathbf{Z}_l(n)) \to 0.$$

Taking continuous Galois cohomology gives an isomorphism of l-primary torsion groups:

$$\frac{H^{2n-1}(\bar{X}, \mathbf{Q}_l/\mathbf{Z}_l(n))^G}{\mathrm{Im}(H^{2n-1}(\bar{X}, \mathbf{Q}_l(n))^G)} \xrightarrow{\sim} H^1(k, H^{2n-1}(\bar{X}, \mathbf{Z}_l(n)))\{l\}.$$

Thus, on torsion, the map ρ_l^n looks like

$$C_{\text{hom}}^n/C_{\text{rat}}^n(X)\{l\} \to \frac{H^{2n-1}\big(\overline{X}, \mathbf{Q}_l/\mathbf{Z}_l(n)\big)^G}{\text{Im}\Big(H^{2n-1}\big(\overline{X}, \mathbf{Q}_l(n)\big)^G\Big)}.$$

If X is proper and geometrically connected over k then the Weil conjectures and a specialization argument [5, Theorem 1] show that $H^{2n-1}(\overline{X}, \mathbf{Q}_l(n))^G = 0$. Hence we get a map

$$C_{\text{hom}}^n(X)/C_{\text{rat}}^n(X)\{l\} \to H^{2n-1}\big(\overline{X}, \mathbf{Q}_l/\mathbf{Z}_l(n)\big)^G.$$

If L is a separably closed field containing k and $X_L = X \times_k L$ we define ρ_l^n for X_L by the formula

$$\big(\rho_l^n\big)_{X_L} = \varinjlim_{\substack{k \subset N \subset L \\ N \text{ finitely generated}}} \big(\rho_l^n\big)_{X_N} : C_{\text{hom}}^n/C_{\text{rat}}^n(X_N)\{l\} \to H^{2n-1}\big(\overline{X}_N, \mathbf{Q}_l/\mathbf{Z}_l(n)\big)\text{Gal}^{(\overline{N}/N)}.$$

$$\|$$
$$H^{2n-1}(X_L, \mathbf{Q}_l/\mathbf{Z}_l(n))$$

This is essentially Bloch's torsion map λ_l^n.

If there is torsion in $H^{2n-1}(\overline{X}, \mathbf{Z}_l(n))$ or $H^{2n}(\overline{X}, \mathbf{Z}_l(n))$ one easily constructs the map by using the exact sequence of G-modules

$$0 \to D \to H^{2n-1}\big(\overline{X}, \mathbf{Q}_l/\mathbf{Z}_l(n)\big) \to T \to 0$$

where T is the torsion subgroup of $H^{2n}(\overline{X}, \mathbf{Z}_l(n))$ and D is the maximal divisible subgroup of $H^{2n-1}(\overline{X}, \mathbf{Q}_l/\mathbf{Z}_l(n))$, and the sequence

$$0 \to T' \to H^{2n-1}\big(\overline{X}, \mathbf{Z}_l(n)\big) \to H^{2n-1}\big(\overline{X}, \mathbf{Z}_l(n)\big)/\text{torsion} \to 0$$

where T' is the torsion subgroup of $H^{2n-1}(\overline{X}, \mathbf{Z}_l(n))$.

References

1. S. Bloch and A. Ogus, *Gersten's conjecture and the homology of schemes*, Ann. Sci. École Norm. Sup. **7** (1974), 181–202.

2. S. Bloch, *Torsion algebraic cycles and a theorem of Roitman*, Compositio Math. **39** (1979), 107–127.

3. _____, *Algebraic cycles and values of L-functions*. I, J. Reine Angew. Math. **350** (1984), 94–108.

4. J.-L. Colliot-Thélène, J.-J. Sansuc, et C. Soulé, *Torsion dans le groupe de Chow de codimension deux*, Duke Math. J. **50** (1983), 763–801.

5. J.-L. Colliot-Thélène and W. Raskind, *K_2-cohomology and the second Chow group*, Math. Ann. **270** (1985), 165–199.

6. S. Lang, *Fundamentals of diophantine geometry*, Springer-Verlag, Berlin and New York, 1983.

7. W. Raskind, *Le théorème de Mordell-Weil faible pour $H^0(X, K_2)/K_2 k$*, C. R. Acad. Sci. Paris (Sér. I) **299** (1984), 241–244.

8. P. Schneider, *Über gewisse Galois cohomologiegruppen*, Math. Z. **168** (1979), 181–205.

9. _____, Thesis, Regensburg, 1980.

10. J.-P. Serre, *Cohomologie galoisienne*, Lecture Notes in Math., vol. 5, Springer-Verlag, Berlin and New York, 1964.

11. C. Soulé, *On higher p-adic regulators*, Algebraic K-theory (Evanston, 1980), Lecture Notes in Math., vol. 854, Springer-Verlag, Berlin and New York, 1981, pp. 372–401.

12. A. A. Suslin, *Torsion in K_2 of fields*, LOMI preprint, Leningrad, 1982.

13. J. Tate, *Relations between K_2 and Galois cohomology*, Invent. Math. **36** (1976), 257–274.

14. SGA 4$\frac{1}{2}$ par P. Deligne, avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J.-L. Verdier, Lecture Notes in Math., vol. 569, Springer-Verlag, Berlin and New York, 1977.

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MASSACHUSETTS 02138

MATHEMATICAL SCIENCES RESEARCH INSTITUTE, 1000 CENTENNIAL DRIVE, BERKELEY, CALIFORNIA 94720